NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.
NOV 1 [illegible] 1958

AFOSR-TN-59-307
ASTIA Document No. AD 213091

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. HT-2

Measure-Theoretic Foundations of Statistical Mechanics

ROBERT M. LEWIS

Contract No. AF 49(638)-341
Project No. 47501

APRIL, 1959

HT-2
c.1

NEW YORK UNIVERSITY

Institute of Mathematical Sciences

Division of Electromagnetic Research

Research Report No. HT-2

Measure-theoretic Foundations of Statistical Mechanics

Robert M. Lewis

Robert M. Lewis

Sidney Borowitz
Acting Project Director

April, 1959

Qualified requestors may obtain copies of this report from the Armed Services Technical Information Agency, Arlington Hall Station, Arlington 12, Virginia. Department of Defense contractors must be established for ASTIA services, or have their 'need to know' certified by the cognizant military agency of their project.

ABSTRACT

The basic formulas of classical equilibrium statistical mechanics are derived from well-known theorems in measure theory and ergodic theory. The method used is a generalization of the methods of Khinchin and Grad and deals with several, in fact a "complete set," of "invariants" or "integrals of the motion." Most of the results are simple corollaries of Birkhoff's ergodic theorem, and since time-averages are used, the whole approach is characterized by an absence of statistical "ensembles" and probability notions. In the course of the development a "generalized temperature" is introduced, and a generalization of the second law of thermodynamics is derived. Formulas for the "microcanonical," "canonical", and "grand canonical" distributions appear as special cases of the general theory.

TABLE OF CONTENTS

	Page
Introduction	1
1. The mathematical theory	3
2. Applications	19
A. Initial value problems	19
B. The generalized temperature	22
C. The generalized "Second Law"	24
D. Classical mechanics	26
E. The grand canonical distribution	31
Appendix I: Proof of Lemma VI	35
Appendix II: Proof of Theorem III	39
References	46

Introduction

The subject of this paper is the foundations of classical equilibrium statistical mechanics. The approach used is similar to that of A.I. Khinchin [5] and H. Grad [2], and is characterized by an absence of statistical 'ensembles'; in fact, probability notions play no role in the theory. Our method generalizes that of Khinchin in two directions. First, following the suggestion of Grad, we consider more than one invariant, or integral of the motion. Second, instead of considering transformations of phase space for which Lebesque measure is invariant, we consider measure-preserving transformations of arbitrary measure spaces.

Section 1, the mathematical theory, may be viewed as a small addition to measure theory and ergodic theory; in fact, most of the results may be considered corollaries of Birkhoff's ergodic theorem. In the beginning of that section, we use the theory of differentiation of set functions to arrive at a formula which is the natural generalization of the 'microcanonical distribution'. This approach, which is very simple, is made possible by the introduction of the notion of a 'complete set of invariant functions'. In order to treat a system in interaction with its environment we introduce a precise definition of 'weak interaction'; and in deriving the generalized form of the 'canonical distribution' we use Grad's definition of a 'large component' rather than Khinchin's approach based on the central limit theorem. In our development, the exponential law of the canonical distribution appears as a natural consequence of an attempt to extend our results by the use of the equivalent measures.

The motivation of the mathematical theory appears in Section 2A. There we discuss physical systems for which an initial value problem is appropriate. We consider the 'state space' of such a system and introduce on this space measures which are preserved by the 'solution operator'. Our approach leads,

in Section 2B, to a generalized notion of temperature and, in Section 2C, to a generalization of the second law of thermodynamics. In Section 2D, we derive the usual results for classical mechanics from our general theory. There (see Lemma VI) we provide a proof of certain assertions which are frequently taken for granted. Finally, in Section 2E, we derive the 'grand canonical distribution' within the framework of the general theory.

As is already evident, we shall make use of the language and basic theorems of abstract measure theory. Hence we begin with a review of a few of the basic definitions:

If Γ is any space, then $(\Gamma, \mathcal{A})$ is called a measurable space if $\mathcal{A}$ is a completely additive class of subsets of Γ, i.e., if $\mathcal{A}$ contains the empty set, ϕ, and is closed under complementation and formation of countable unions. The elements of $\mathcal{A}$ are called measurable sets. If $(\Gamma, \mathcal{A})$ and $(\Gamma', \mathcal{A}')$ are measurable spaces, then a single-valued function f from Γ into Γ' is a measurable function or measurable transformation if the inverse image of every set $A' \in \mathcal{A}'$ is a set $A \in \mathcal{A}$. For real-valued functions, Γ' is the real line and, unless otherwise specified, $\mathcal{A}'$ is taken to be the class of Borel sets, the smallest completely additive class of sets containing all the open sets. If $(\Gamma, \mathcal{A})$ is a measurable space then $(\Gamma, \mathcal{A}, m)$ is a measure space if m is a measure function, i.e., if m is a non-negative real-valued set function defined on $\mathcal{A}$ such that

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$$

for all sequences $\{A_n\}$ of disjoint sets of $\mathcal{A}$ and $m(\phi) = 0$. Every measure space is assumed to be σ-finite; i.e., Γ is the union of a countable collection of sets of finite measure.

Let $(\Gamma_\nu, \mathcal{A}_\nu, m_\nu)$ be measure spaces for $\nu = 1,\ldots,n$. Let A_ν be a subset of Γ_ν and let γ_ν denote a point (element) of Γ_ν. The product set $A_1 \times \cdots \times A_n$ is the set of all ordered n-tuples $(\gamma_1,\ldots,\gamma_n)$ where $\gamma_\nu \in A_\nu$. If each A_ν is a measurable set of finite measure then $A_1 \times \cdots \times A_n$ is called a rectangle. The smallest completely additive class of subsets of $\Gamma_1 \times \cdots \times \Gamma_n$ containing all rectangles $A_1 \times \cdots \times A_n$ is called the product class and is denoted $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$. It can be shown that there exists a unique measure m on $(\Gamma_1 \times \cdots \times \Gamma_n, \mathcal{A}_1 \times \cdots \mathcal{A}_n)$ such that for every rectangle, $m(A_1 \times \cdots \times A_n) = \prod_{\nu=1}^{n} m_\nu(A_\nu)$. m is called the product measure and is denoted by $m_1 \times \cdots \times m_n$. $(\Gamma_1 \times \cdots \times \Gamma_n, \mathcal{A}_1 \times \cdots \times \mathcal{A}_n, m_1 \times \cdots \times m_n)$ is a σ-finite measure space.

If $(\Gamma, \mathcal{A}, m)$ is a measure space, a measurable transformation T from Γ into itself is said to be measure-preserving if* $m(T^{-1}A) = m(A)$ for every $A \in \mathcal{A}$, and a real-valued measurable function $f(\gamma)$ defined on Γ is said to be an invariant function, or an invariant, if $f(T\gamma) = f(\gamma)$ a.e. (i.e., 'almost everywhere' - except on a set of m-measure zero). If T_t is a family of measure-preserving transformations depending on the parameter t, then f is an invariant function relative to the family if there exists a set C of measure zero such that for $\gamma \in -C$, $f(T_t\gamma) = f(\gamma)$ for all t. -C denotes the complement of C.

1. The mathematical theory

Let $(\Gamma, \mathcal{A}, m)$ be a measure space, and let T_t be a family of transformations of Γ into itself which are measure-preserving for $t \in \mathcal{T}$, where $\mathcal{T}$ denotes the real semi-axis $0 \le t < \infty$. Let $\mathcal{M}$ denote the class of Lebesgue-measurable subsets of $\mathcal{T}$. We shall say that $(\Gamma, \mathcal{A}, m, T_t)$ is a measure-

---

* $T^{-1}A$ denotes the set of all elements γ such that $T\gamma$ is an element in the set A.

preserving space whenever $T_{s+t} = T_s T_t$ for each t and s in $\mathcal{T}$ and $T_t\gamma$ is a measurable function of t and γ; i.e., $T_t\gamma$ is a measurable transformation from $(\mathcal{T}\times\Gamma, \mathcal{M}\times\mathcal{A})$ into $(\Gamma,\mathcal{A})$.

Our work in this section is based mainly on the following theorem of Birkhoff, a proof of which is presented in [4].

Ergodic Theorem

If $(\Gamma,\mathcal{A},m,T_t)$ is a measure-preserving space and $f(\gamma)$ is integrable on $(\Gamma,\mathcal{A},m)$ then the limit

$$(1) \qquad f^*(\gamma) = \lim_{L\to\infty} \frac{1}{L}\int_0^L f(T_t\gamma)dt$$

exists a.e. (The integral in (1) is a Lebesgue integral.) $f^*(\gamma)$ is integrable and invariant relative to the family T_t. If $m(\Gamma)$ is finite then

$$(2) \qquad \int f^*(\gamma)dm = \int f(\gamma)dm.$$

Given a measure-preserving space $(\Gamma,\mathcal{A},m,T_t)$, a set of real-valued measurable functions $y_1,\ldots,y_k$ will be called a complete set of invariant functions, and the vector $y(\gamma) = \{y_1,\ldots,y_k\}$ will be called a complete invariant vector if each of the functions $y_j(\gamma)$ is an invariant relative to the family T_t, and if every real-valued measurable invariant function $z(\gamma)$ is a measurable function of $\{y_1,\ldots,y_k\}$ a.e. (If R^k denotes k-dimensional euclidean space, and B^k denotes the class of Borel sets in R^k, a Borel measurable function is one which is measurable from (R^k,B^k) to (R^1,B^1). We require that $z(\gamma) = Z[y(\gamma)]$ a.e., where Z is Borel measurable.) For convenience we shall say that $(\Gamma,\mathcal{A},m,T_t,y)$ is a complete space whenever $(\Gamma,\mathcal{A},m,T_t)$ is a measure-preserving space, and y is a complete invariant vector.

If $(\Gamma, \mathcal{A}, m, T_t, y)$ is a complete space and f is integrable, then according to the Ergodic Theorem

$$f^*(\gamma) = F\left[y(\gamma)\right] \text{ a.e.} \tag{3}$$

Thus we can evaluate f^* if we can evaluate F. In what follows we shall show how this can be done.

Given any Borel set B in B^k, let $A = y^{-1}(B) = \left\{\gamma \ni y(\gamma) \in B\right\}^*$. Then $A \in \mathcal{A}$ and for all $t \geq 0$, T_t maps A- C into itself where C is a set of measure zero. Accordingly we may apply the Ergodic theorem using as the basic space not Γ, but A-C. It follows from (2) that if $m(A) < \infty$

$$\int_{A-C} f^* \, dm = \int_{A-C} f \, dm \tag{4}$$

and since $m(C) = 0$

$$\int_A f^* dm = \int_A f \, dm. \tag{5}$$

Let M be the measure induced in (R^k, B^k) by y and m, i.e., for any $B \in B^k$

$$M(B) = m\left[y^{-1}(B)\right]. \tag{6}$$

It can be shown that if G is a Borel measurable function such that $G\left[y(\gamma)\right]$ is integrable on $(\Gamma, \mathcal{A}, m)$ then G is integrable on (R^k, B^k, M) and

$$\int_{y^{-1}(B)} G\left[y(\gamma)\right] dm = \int_B G(y) dM. \tag{7}$$

It follows now from (3), (5), and (7) that for any $B \in B^k$ such that $m\left[y^{-1}(B)\right] < \infty$

$$\int_B F(y) dM = \int_{y^{-1}(B)} F\left[y(\gamma)\right] dm = \int_{y^{-1}(B)} f^* dm = \int_{y^{-1}(B)} f \, dm. \tag{8}$$

---

* The expression $\left\{\gamma \ni y(\gamma) \in B\right\}$ means: The set of all elements γ such that $y(\gamma)$ is an element of the set B.

Integrability of F follows from that of f^*.

In order to evaluate F we shall apply a generalization of the well-known theorem which asserts that the derivative of an indefinite integral is equal a.e. to the integrand. We state, without proof, the following lemma, which follows easily from the theory of differentiation of set functions with respect to 'nets'. A full discussion of the theory appears in [6].

Lemma I

Let B^k be the class of Borel sets in R^k and let (R^k,B^k,M) be a measure space such that M assigns finite values to finite intervals. Let F be integrable on (R^k,B^k,M). Then for almost every $x \in R^k$ there exists a sequence of half-open intervals

$$I_j(x) = \left\{ y \ni c^j < y-x \leq d^j \right\} = \left\{ y_1,\ldots,y_k \ni c_\nu^j < y_\nu - x_\nu \leq d_\nu^j;\ \nu=1,\ldots,k \right\}$$

such that for each $j = 1,2,\ldots; c^j < 0 \leq d^j$; $\lim_{j\to\infty} (d^j - c^j) = 0$; $I_{j+1}(x) \subset I_j(x)$; and

$$(9) \qquad \lim_{j\to\infty} \frac{\int_{I_j(x)} F\,dM}{M\left[I_j(x)\right]} = F(x).$$

Now from (8) and (9) we have

$$(10) \qquad F(x) = \lim_{j\to\infty} \frac{\int_{y^{-1}\left[I_j(x)\right]} f\,dm}{m\left\{y^{-1}\left[I_j(x)\right]\right\}} \quad \text{a.e.}$$

From (3) we see that f^* has the same value for all γ for which $y(\gamma)$ has the same value. If $y^{-1}(x)$ denotes the set $\left\{ \gamma \ni y(\gamma) = x \right\}$ it is natural to write $f^*\left[y^{-1}(x)\right]$ for the value of $f^*(\gamma)$ common to all γ in $y^{-1}(x)$. Summarizing our results, we have:

Theorem I

Let $(\Gamma, \mathcal{A}, m, T_t, y)$ be a complete space such that for every finite interval I in R^k, $m\left[y^{-1}(I)\right]$ is finite. Let f be integrable on $(\Gamma, \mathcal{A}, m)$. Then

$$f^*(\gamma) = \lim_{L\to\infty} \frac{1}{L} \int_0^L f(T_t\gamma)dt$$

exists a.e. There exist sequences c^j and d^j of points in R^k which depend on x, for which $c^j < 0 \leq d^j$, $\lim_{j\to\infty} (d^j - c^j) = 0$, and

$$(11) \qquad f^*\left[y^{-1}(x)\right] = \lim_{j\to\infty} \frac{\int_{\{\gamma \ni c^j < y(\gamma)-x \leq d^j\}} f\, dm}{m\left\{\gamma \ni c^j < y(\gamma)-x \leq d^j\right\}} \quad \text{a.e.}$$

Theorem I will be applied later to describe the behavior of isolated physical systems. Equation (11) is a generalization of what is sometimes called the 'microcanonical distribution'. We turn now to the derivation of a formula appropriate to a physical system in interaction with its environment.

Let $y' = \left\{y_1', \ldots, y_k'\right\}$ and $y'' = \left\{y_1'', \ldots, y_k''\right\}$ be vector functions with values in R^k. The complete spaces $(\Gamma', \mathcal{A}', m', T_t', y')$ and $(\Gamma'', \mathcal{A}'', m'', T_t'', y'')$ will be said to be in weak interaction if there exists a complete space $(\Gamma, \mathcal{A}, m, T_t, y)$ where $\Gamma = \Gamma' \times \Gamma''$; $\mathcal{A} = \mathcal{A}' \times \mathcal{A}''$; $m = m' \times m''$; and $y(\gamma) = y(\gamma', \gamma'') = \left\{y_1, \ldots, y_k\right\} = y'(\gamma') + y''(\gamma'')$.* If $m\left[y^{-1}(I)\right]$ is finite for every finite interval I in R^k and if $f(\gamma')$ is integrable on $(\Gamma', \mathcal{A}', m')$, then f is integrable on $(\Gamma, \mathcal{A}, m)$ and we may apply Theorem I. It follows that

$$(12) \qquad f^*(\gamma) = \lim_{L\to\infty} \frac{1}{L} \int_0^L f(T_t\gamma)dt$$

---

* In the applications we shall see that it may be possible to satisfy the equation $y = y' + y''$ only in an approximate sense.

exists a.e., and $f^*\left[\underline{y}^{-1}(x)\right]$ is given by (11).

We should note that although we have taken f to be a function of γ' alone, f^* is a function of $\gamma = (\gamma', \gamma'')$. In (12), $f(T_t\gamma)$ means $f(\beta)$ where β is the first component of $T_t\gamma$. According to our assumptions, $dm = dm'dm''$ in (11) and we may use Fubini's theorem to carry out the integration with respect to γ''. The result is

$$(13) \qquad f^*\left[\underline{y}^{-1}(x)\right] = \lim_{j\to\infty} \frac{\int f(\gamma')m''\left\{\gamma'' \ni c^j < y'(\gamma')+y''(\gamma'')-x \le d^j\right\} dm'}{m\left\{\gamma \ni c^j < y(\gamma) - x \le d^j\right\}} .$$

In the applications, the system of interest will be represented by Γ', and Γ'' will represent the environment, which is assumed to be, in some sense, 'large'. Before giving a precise definition of 'large', let us note that whenever $(\Gamma', \mathcal{A}', m', T_t', y')$ and $(\Gamma'', \mathcal{A}'', m'', T_t'', y'')$ are complete spaces in weak interaction it is possible to find new measures p', p" such that $(\Gamma', \mathcal{A}', p', T_t', y')$ and $(\Gamma'', \mathcal{A}'', p'', T_t'', y'')$ are complete spaces in weak interaction and $p'(\Gamma') = p''(\Gamma'') = 1$. We shall see shortly how this can be done*.

---

* In case $m'(\Gamma') < \infty$ and $m''(\Gamma'') < \infty$ we need only set $p' = m'/m'(\Gamma')$, $p'' = m''/m''(\Gamma'')$.

Definition : $(\Gamma'', \mathcal{A}'', m'', T_t'', y'')$ is large compared to $(\Gamma', \mathcal{A}', m', T_t', y')$, or in symbols $(\Gamma'', \mathcal{A}'', m'', T_t'', y'') \gg (\Gamma', \mathcal{A}', m', T_t', y')$ if the two complete spaces are in weak interaction and the following condition is satisfied: Whenever p'', p' are measures such that $(\Gamma'', \mathcal{A}'', p'', T_t'', t'')$, $(\Gamma', \mathcal{A}', p', T_t', y')$ are complete spaces in weak interaction and $p'(\Gamma') = p''(\Gamma'') = 1$, there exists a vector λ in R^k (depending on p' and p'') such that for $c < 0 \leq d$ and d-c sufficiently small

$$(14) \qquad g(y') = p''\left\{\gamma'' \ni c < y''(\gamma'') + y' - \lambda \leq d\right\} = \text{const.}$$

for almost all values of y' (i.e., except for y' in some Borel set B such that $p'\left\{\gamma' \ni y'(\gamma') \in B\right\} = 0$).*

---

*This definition is a generalized form of the definition of a 'large component' introduced by Grad (see [2], p. 469). Its physical interpretation is discussed further in the next section. Grad's approach is certainly much simpler than that of Khinchin[5], who postulates that the environment consists of a large number, n-1, of weakly interacting systems more or less similar to the system of interest. In that case the quantities p'' and p appearing in (17) may each be interpreted as the probability that the sum of a large number of independent random variables falls within a certain interval. If we then choose λ to be the mean of the distribution of the sum, and if we further require that the distributions be continuous (i.e., representable by density functions in R^k) then we can apply the central limit theorem and obtain the result (19) asymptotically for $n \to \infty$.

In the applications we shall not require that (14) hold exactly; rather we shall suppose that there is some parameter, ℓ, which measures how 'large' the environment is, and (14) will hold asymptotically for $\ell \to \infty$. Consequently (18), (19) and equations obtained from (19) will be viewed as asymptotic results.

Now suppose $(\Gamma'', \mathcal{A}'', m'', T''_t, y'') \gg (\Gamma', \mathcal{A}', m', T'_t, y')$, and p'', p' are measures with the above properties. Let $p = p' \times p''$, and let us use the abbreviated notation

$$p''\left[y'(\gamma')\right] = p''\left\{\gamma'' \ni c^j < y'(\gamma') + y''(\gamma'') - \lambda \leq d^j\right\}, \tag{15}$$

$$p = p\left\{\gamma \ni c^j < y(\gamma) - \lambda \leq d^j\right\}. \tag{16}$$

Then from (13)

$$f^*\left[y^{-1}(\lambda)\right] = \lim_{j\to\infty} \frac{\int f(\gamma')p''\left[y'(\gamma')\right]dp'}{p}. \tag{17}$$

If j is sufficiently large, then $d^j - c^j$ is as small as we wish and $p''\left[y'(\gamma')\right] = p''$ a.e. where p'' is a constant. Thus

$$\int f(\gamma')p''\left[y'(\gamma')\right]dp' = \int f(\gamma')dp' \cdot p'' = \int f(\gamma')dp' \cdot \int p''\left[y'(\gamma')\right]dp' \tag{18}$$

$$= \int f(\gamma')dp' \cdot p$$

and

$$f^*\left[y^{-1}(\lambda)\right] = \int f(\gamma')dp'. \tag{19}$$

Equation (19) is a simple formula for $f^*(\gamma)$, but it holds only for those points γ such that $y(\gamma) = \lambda$. In order to obtain a simple formula for $f^*\left[y^{-1}(x)\right]$ for values of x not equal to λ we shall introduce new measures. In this connection, the following lemma will be used:

Lemma II : Let T_t be a family of measure-preserving transformations with respect to $(\Gamma, \mathcal{A}, m)$. Let $\phi(\gamma)$ be non-negative and integrable on $(\Gamma, \mathcal{A}, m)$. For each $A \in \mathcal{A}$, let $q(A) = \int_A \phi(\gamma)dm^*$. Then T_t is a family of measure-

---

Then q is a measure on $(\Gamma, \mathcal{A})$.

preserving transformations with respect to $(\Gamma, \mathcal{A}, q)$ if and only if ϕ is an invariant relative to the family T_t and $(\Gamma, \mathcal{A}, m)$.

Proof: By definition $m(T_t^{-1}A) = m(A)$ for $A \in \mathcal{A}$.

1. Let ϕ be an invariant: $\phi(T_t\gamma) = \phi(\gamma)$ a.e. Then $q(T_t^{-1}A) = \int_{T_t^{-1}A} \phi(\gamma)dm = \int_A \phi(T_t\gamma)dm = \int_A \phi(\gamma)dm = q(A)$. Thus T_t preserves q.

2. Let T_t preserve q: $q(T_t^{-1}A) = q(A)$. Then $\int_A \phi(\gamma)dm = q(A) = q(T_t^{-1}A) = \int_A \phi(T_t\gamma)dm$ for all A . Hence $\phi(\gamma) = \phi(T_t\gamma)$ a.e., i.e., ϕ is an invariant.

Now suppose $(\Gamma', \mathcal{A}', m', T_t', y')$, $(\Gamma'', \mathcal{A}'', m'', T_t'', y'')$ are complete spaces in weak interaction. On each space we define a new measure

$$q'(A') = \int_{A'} \phi'(\gamma')dm', \qquad q''(A'') = \int_{A''} \phi''(\gamma'')dm''. \tag{20}$$

By the lemma, $(\Gamma', \mathcal{A}', q', T_t', y')$ will be a complete space if and only if ϕ' is an invariant with respect to $(\Gamma', \mathcal{A}', m', T_t', y')$. Since y' is a complete invariant vector, this will be true if and only if $\phi'(\gamma') = \Phi'\left[y'(\gamma')\right]$, where Φ' is a Borel-measurable function on R^k. Similarly $(\Gamma'', \mathcal{A}'', q'', T_t'', y'')$ will be a complete space if and only if $\phi''(\gamma'') = \Phi''\left[y''(\gamma'')\right]$. Now let $q = q' \times q''$ be the product measure on $(\Gamma, \mathcal{A})$. For any rectangle

$$q(A' \times A'') = q'(A')q''(A'') = \int_{A'} \Phi'[y']dm' \int_{A''} \Phi''[y'']dm'' \tag{21}$$

$$= \int_{A'}\int_{A''} \Phi'[y']\Phi''[y'']dm'dm'' = \int_{A' \times A''} \Phi'[y']\,\Phi''[y'']dm.$$

It follows that for any $A \in \mathcal{A}$, $q(A) = \int_A \overline{\Phi}'[y'] \, \overline{\Phi}''[y'']dm$. Now, as before, $(\Gamma, \mathcal{A}, q, T_t, y)$ will be a complete space if and only if

$$\overline{\Phi}'[y'] \, \overline{\Phi}''[y''] = \overline{\Phi}[y] = \overline{\Phi}[y' + y''] \tag{22}$$

where again $\overline{\Phi}$ is Borel-measurable on R^k. (22) is a functional equation for the functions $\overline{\Phi}'$, $\overline{\Phi}''$, $\overline{\Phi}$ which has the solutions $\overline{\Phi}'[y'] = c'e^{-\alpha \cdot y'}$; $\overline{\Phi}''[y''] = c''e^{-\alpha \cdot y''}$; $\overline{\Phi}[y] = c'c''e^{-\alpha \cdot y}$; where $\alpha = \alpha_1, \ldots, \alpha_k$ is an arbitrary vector in R^k, c' and c'' are arbitrary non-negative real numbers, and $\alpha \cdot y = \sum_{j=1}^{k} \alpha_j y_j$, etc. For convenience, we shall refer to measures q and m which are related by equations of the form $q(A) = \int_A ce^{-\alpha \cdot y(\gamma)}dm$, $(A \in \mathcal{A})$, as exponentially equivalent measures.

Now, let us assume that the complete spaces $(\Gamma', \mathcal{A}', m', T_t', y')$ and $(\Gamma'', \mathcal{A}'', m'', T_t'', y'')$ are in weak interaction, where m' and m'' are not assumed to be normalizable, i.e., we do not assume that $m'(\Gamma')$ and $m''(\Gamma'')$ are finite. We now consider the class of normalizable measures which are exponentially equivalent to m' and m'': Let D' and D'' be subsets of R^k consisting of all vectors α such that

$$z'(\alpha) = \int e^{-\alpha \cdot y'}dm' \quad \text{and} \quad z''(\alpha) = \int e^{-\alpha \cdot y''}dm'' \tag{23}$$

are, respectively, finite. We assume that $D'' \supset D'$ and that D' is a non-empty set. For each α in D' set

$$z(\alpha) = z'(\alpha)z''(\alpha). \tag{24}$$

By Fubini's theorem

$$z(\alpha) = \int e^{-\alpha \cdot (y' + y'')}dm'dm'' = \int e^{-\alpha \cdot y} \, dm. \tag{25}$$

Now for $A' \in \mathcal{A}'$ and $A'' \in \mathcal{A}''$ set

$$(26) \qquad p'(A') = \left[z'(\alpha)\right]^{-1} \int_{A'} e^{-\alpha \cdot y'} dm'; \qquad p''(A'') = \left[z''(\alpha)\right]^{-1} \int_{A''} e^{-\alpha \cdot y''} dm''.$$

Then if $p = p' \times p''$,

$$(27) \qquad p(A) = \left[z(\alpha)\right]^{-1} \int_{A} e^{-\alpha \cdot y} dm$$

for every set $A \in \mathcal{A}$. Of course, the measures p', p'', p depend parametrically on α.

Now $p'(\Gamma') = p''(\Gamma'') = 1$, and we may apply (19). We summarize our results in the following theorem.

Theorem II

Let $(\Gamma'', \mathcal{A}'', m'', T_t'', y'') \gg (\Gamma', \mathcal{A}', m', T_t', y')$. Let D', D'', and $z'(\alpha)$ be defined as above and let $D'' \supset D'$ where D' is a non-empty set. Let $f(\gamma')e^{-\alpha \cdot y'(\gamma')}$ be integrable on $(\Gamma', \mathcal{A}', m')$ for each α in D'. Then

$$(28) \qquad f^*(\gamma) = \lim_{L \to \infty} \frac{1}{L} \int_0^L f(T_t \gamma) dt$$

exists a.e. with respect to $m' \times m''$, and for each α in D' there exists a vector λ_α in R^k such that

$$(29) \qquad f^*\left[y^{-1}(\lambda_\alpha)\right] = \bar{f}(\alpha) = \left[z'(\alpha)\right]^{-1} \int f(\gamma')e^{-\alpha \cdot y'(\gamma')} dm'.$$

Theorem II, and in particular formula (29), is our main mathematical result and will form the basis for the physical applications. However, before proceeding to these applications we shall take a closer look at some of the mathematical concepts, and derive a corollary to Theorem II. We begin with some definitions.

Let (R^k, B^k, M) be a measure space. B^k denotes the class of Borel sets in R^k. Let E be any open half-space in R^k such that $M(E) = 0$. Let F be the union of all such open half-spaces of measure zero. F is open, and its complement H is closed. Since H is the intersection of all closed half-spaces of 'full measure' M, H will be called the closed convex hull of M. Let $\tilde{H}$ be the complement of the closure of F. Then $\tilde{H}$ is the largest open subset of H. $\tilde{H}$ will be called the open convex hull of M.

Lemma III

If $M(R^k) < \infty$, and

$$\bar{x} = \frac{\int x \, dM}{M(R^k)},$$

then $\bar{x}$ is a point in the closed convex hull H of M. (x denotes a point $\{x_1, \ldots, x_k\}$ in R^k.)

Proof: Suppose $\bar{x} \notin H$. Then $\bar{x} \in F$; hence $\bar{x} \in E$ where E is an open half-space of measure zero. By a rotation of coordinates we can assume that E is the open half-space $a < x_1$ where a is a real constant. Thus $a < \bar{x}_1$. But

$$\int x_1 \, dM = \int_{x_1 \leq a} x_1 \, dM \leq aM(R^k). \tag{30}$$

Hence $\bar{x}_1 \leq a$. This is a contradiction.

We shall make use of this lemma a little later, but now we shall use the above definitions to examine more closely the notion of a complete set of invariants. If $(\Gamma, \mathcal{A}, m, T_t, y)$ is a complete space, the invariant functions $y_1, \ldots, y_k$ in a sense span the space of all invariant functions. The set $y_1, \ldots, y_k$ can always be enlarged by adding invariant functions and remains a complete set. But under what conditions can we reduce the number of functions

and still have a complete set? The following lemma is an answer to this question.

Lemma IV

Let $(\Gamma, \mathcal{A}, m, T_t, y)$ be a complete space, where $y = \{y_1, \ldots, y_k\}$. Let M be the measure induced in (R^k, B^k) by y and m, i.e., $M(B) = m\left[y^{-1}(B)\right]$ for $B \in B^k$, and let $\tilde{H}$ be the open convex hull of M. If $\tilde{H}$ is empty then there exists a complete set of invariants $\{w_1, \ldots, w_{k-1}\}$.

Proof: If $\tilde{H}$ is empty, it is easy to see that there exists a hyperplane $L \subset R^k$ of 'full measure'; i.e., there exists a set L of the form

$$L = \left\{ y \ni \sum_{i=1}^{k} a_i(y_i - \mu_i) = 0 \right\} \tag{31}$$

such that $M(-L) = 0$. The a_i and μ_i are real constants and at least one a_i is non-zero. Hence we can define a non-singular matrix (a_{ij}); i,j = 1,...,k; where $a_{ik} = a_i$; i=1,...,k. Let us introduce the linear transformation

$$w_\nu = \sum_{i=1}^{k} a_{i\nu}(y_i - \mu_i); \quad \nu = 1, \ldots, k. \tag{32}$$

The w_ν are invariants. Now $y^{-1}(L) = \{\gamma \ni w_k(\gamma) = 0\}$, hence $y^{-1}(-L) = \{\gamma \ni w_k(\gamma) \neq 0\}$ and $m[y^{-1}(-L)] = M(-L) = 0$. Thus $w_k(\gamma) = 0$ a.e.

If $z(\gamma)$ is any invariant then $z(\gamma) = Z[y(\gamma)] = G[w_1, \ldots, w_k]$ a.e., where Z and G are Borel-measurable on R^k. Thus $z(\gamma) = G[w_1, \ldots, w_{k-1}, 0] = J[w_1, \ldots, w_{k-1}]$ a.e., where J is Borel-measurable on R^{k-1}. Thus $w_1, \ldots, w_{k-1}$ is a complete set of invariants.

We see from this lemma that if $\tilde{H}$ is empty we can always obtain a smaller set of invariants which is complete. If the new $\tilde{H}$ is empty we can repeat the process, and so on, until eventually we reach a complete set of invariants for which $\tilde{H}$ is non-empty, except in the case where the single function $y(\gamma)$ is a complete set of invariants and the corresponding $\tilde{H}$ is empty. In this case the closed convex hull H consists of a single point c in R^1 and $y = c$ a.e.

A measure-preserving space will be called indecomposable* if the only invariant functions are those which are constant a.e. Whenever $\tilde{H}$ is a non-empty, the complete set of invariants $y_1,\ldots,y_k$ will be said to be non-singular. Thus we have

Lemma V

Let $(\Gamma, \mathcal{A}, m, T_t, \{y_1,\ldots,y_k\})$ be a complete space. Then either $(\Gamma, \mathcal{A}, m, T_t)$ is indecomposable or there exists a non-singular complete set of invariants $\{z_1,\ldots,z_j\}$, where $j \leq k$.

Let us consider, for a moment, the indecomposable case: $(\Gamma, \mathcal{A}, m, T_t, y)$ is a complete space, and $y = \text{const.a.e.}$ If $m(\Gamma) < \infty$ we can apply the Ergodic Theorem immediately and from (2)

$$(32) \qquad f^*(\gamma) = \lim_{L\to\infty} \frac{1}{L}\int_0^L f(T_t\gamma)dt = \left[m(\Gamma)\right]^{-1}\int f(\gamma)dm = \text{const.} \quad \text{a.e.}$$

If the complete spaces $(\Gamma', \mathcal{A}', m', T_t', y')$ and $(\Gamma'', \mathcal{A}'', m'', T_t'', y'')$ are in weak interaction, and both y' and y'' are constant a.e., then $y = y' + y''$ is a constant a.e. If $m'(\Gamma')$ and $m''(\Gamma'')$ are finite, then $m(\Gamma) = m' \times m''(\Gamma' \times \Gamma'')$ is finite, and if $f(\gamma')$ is integrable on $(\Gamma', \mathcal{A}', m')$ then

$$(33) \qquad f^*(\gamma) = \lim_{L\to\infty} \frac{1}{L}\int_0^L f(T_t\gamma)dt = \left[m(\Gamma)\right]^{-1}\int f(\gamma')dm = \left[m'(\Gamma')\right]^{-1}\int f(\gamma')dm'.$$

We see from (32) and (33) that in the indecomposable case the calculations are essentially trivial. We shall not make use of these results in the applications.

The importance of Lemma V is that, except for the indecomposable case we may always assume that a complete space has a non-singular complete set of invariants. We shall make use of this fact in what follows.

---

* This is in agreement with the usual usage. (See e.g. [4], p. 25.) The terms 'ergodic' and 'metrically transitive' are also used.

Let us return now to Theorem II. The results (28) and (29) hold, of course, when $f(\gamma')$ is the vector function $y'(\gamma')$. Thus

$$(34) \qquad y'^{*}\left[\bar{y}^{-1}(\lambda_{\alpha})\right] = \bar{y}'(\alpha) = \left[z'(\alpha)\right]^{-1}\int y'(\gamma')e^{-\alpha\cdot y'(\gamma')}dm' = \int y'(\gamma')dp'$$

where p' is defined by (26). Now let M' be the measure induced in (R^k,B^k) by m' and y'. Similarly let P' be the measure induced by p' and y'. Since M' and P' have the same sets of measure zero they have the same open and closed convex hulls $\tilde{H}$ and H. From (34)

$$(35) \qquad \bar{y}'(\alpha) = \int_{R^k} x\, dP'$$

and $P'(R^k) = 1$. It follows from Lemma III that $\bar{y}'(\alpha)$ lies in H. Thus $\bar{y}'(\alpha)$ is a single-valued function from D' into H. We will show that this function has a single-valued inverse on $\tilde{H}$, provided D' is an open connected set. The following corollary is a generalization of a theorem of Grad [2], which in turn is a generalization of a theorem of Khinchin [5].

Corollary to Theorem II:

Let $\tilde{H}$ be the open convex hull of the measure M' induced in (R^k,B^k) by m' and y'. Let D' be an open connected set. Then the function $\bar{y}'(\alpha)$ defined by (34) has a single-valued inverse on $\tilde{H}$, i.e., to any $c = \{c_1,\ldots,c_k\}$ in $\tilde{H}$, there exists a unique α in D' such that $\bar{y}'(\alpha) = c$. Thus for $\bar{y}'$ in $\tilde{H}$, $\bar{f} = \bar{f}[\alpha(\bar{y}')]$. $\bar{f}(\alpha)$ is given explicitly by (29) and $\alpha(\bar{y}')$ is defined implicitly by (34).*

Proof: If v is any non-zero vector in R^k, the equation $v\cdot(c-y') = 0$ defines a hyperplane through c and perpendicular to v. Both open half-spaces bounded by this hyperplane must be of positive M'-measure; for if one were not, then every neighborhood of c would intersect F, the union of all open half-spaces of measure zero, and c would be in the closure of F, whereas it is in the complementary set. Now let

$$(36) \qquad \psi(\alpha) = e^{\alpha\cdot c}\, z'(\alpha) = \int e^{\alpha\cdot(c-y')}\, dm',$$

---

*Note that except for the indecomposable case, we can assume that $\tilde{H}$ is non-empty. This assures us that the corollary is not an empty statement.

$$(37) \qquad \psi_{ij} = \frac{\partial^2 \psi}{\partial \alpha_i \partial \alpha_j} = \int \left[c_i - y_i'\right]\left[c_j - y_j'\right] e^{\alpha \cdot (c-y')} dm'.$$

It follows from the above discussion that

$$(38) \qquad 0 < \int \left[v \cdot (c-y')\right]^2 e^{\alpha \cdot (c-y')} dm' = \sum_{i,j} \psi_{ij} v_i v_j .$$

Hence (ψ_{ij}) is a positive definite matrix and $\psi(\alpha)$ is a 'convex' function.

We will show that $\psi(\alpha) \to \infty$ as $\alpha \to \infty$ in any direction from an arbitrary point α_o in D'. Choose any unit vector α_1 $(\alpha_1 \cdot \alpha_1 = 1)$ and set $\alpha = \alpha_o + t\alpha_1$, $t \geq 0$. As t increases, either (A) α reaches the boundary of D' or (B) $t \to \infty$. In case (A), $z'(\alpha)$ becomes unbounded for finite α, hence $\psi(\alpha)$ becomes unbounded. In case (B) since $\tilde{H}$ is open, there exists an $\epsilon > 0$ such that the point $c - \epsilon\alpha_1$ is in $\tilde{H}$. If we consider the hyperplane through that point, $\alpha_1 \cdot (c - \epsilon\alpha_1 - y') = \alpha_1 \cdot (c-y') - \epsilon = 0$, the half-space $\alpha_1 \cdot (c-y') - \epsilon > 0$ must be of positive M'-measure. Therefore

$$m' \left\{ \gamma' \ni \alpha_1 \cdot \left[c - y'(\gamma')\right] > \epsilon \right\} > 0.$$

Thus

$$(39) \quad \psi(\alpha) = \int e^{(\alpha_o + t\alpha_1)\cdot(c-y')} dm' \geq e^{\alpha_o \cdot c} \int_{\left\{\gamma' \ni \alpha_1 \cdot [c-y'(\gamma')] > \epsilon\right\}} e^{t\alpha_1 \cdot [c-y'(\gamma')] - \alpha_o \cdot y'} dm'$$

$$\geq e^{\alpha_o \cdot c} e^{t\epsilon} \int_{\left\{\gamma' \ni \alpha_1 \cdot [c-y'(\gamma')] > \epsilon\right\}} e^{-\alpha_o \cdot y'} dm' .$$

We see that $\psi(\alpha) \geq a_1 e^{t\epsilon}$ where $a_1 > 0$ and $\epsilon > 0$. Therefore $\psi(\alpha) \to \infty$ as $t \to \infty$.

It follows that $\psi(\alpha)$ has a minimum at some point α in D'. Since D' is connected, α is unique. The point α is also a minimum for $\log \psi = \alpha \cdot c + \log z'(\alpha)$. Hence, at this point $\frac{\partial}{\partial \alpha_i} \log \psi = c_i - (z')^{-1} \int y_i' e^{-\alpha \cdot y'} dm' = 0$, $i = 1, \ldots, k$; i.e. $c = [z'(\alpha)]^{-1} \int y'(\gamma') e^{-\alpha \cdot y'(\gamma')} dm' = \bar{y}'(\alpha)$.

2. Applications

A. Initial value problems

The mathematical theory we have developed will be applied to the study of the behavior of a system of the type for which an initial valve problem is appropriate. This problem is to determine for all times, t, the "state" of the system when the state at $t = 0$ is specified. The state is usually given by a set of numbers or functions. We shall denote the states, whatever they may be, by γ, and the state space (the set of all states) by Γ. If we assume the existence and uniqueness of the solution of the initial value problem for all $t \geq 0$, then we may define a one-parameter family of transformations T_t of Γ into itself depending on the non-negative parameter T : For any $\gamma \in \Gamma$ and any $t \geq 0$, $T_t\gamma$ is the state at time t which is the solution of the initial value problem corresponding to the state γ at $t = 0$. T_t is sometimes called the solution operator . It is immediately apparent that for each non-negative t and s, $T_{s+t} = T_s T_t$. In general we are unable actually to solve the initial value problem, but we are interested in obtaining information about the system without knowing the transformations T_t explicitly.

If we can find a completely additive class $\mathcal{A}$ of subsets of Γ, a measure function m, and a complete invariant vector y, such that $(\Gamma, \mathcal{A}, m, T_t, y)$ is a complete space and $m[y^{-1}(I)]$ is finite for every finite interval I in R^k, then we can apply theorem I: If $f(\gamma)$ is integrable on $(\Gamma; a, m)$ then the "infinite time average"

$$(40) \qquad f^*(\gamma) = \lim_{L\to\infty} \frac{1}{L} \int_0^L f(T_t\gamma)dt$$

exists a.e. and f^* is given by the formula (11). If $f(\gamma)$ represents a physically observable quantity associated with the system in the state γ, we may assume that a measurement of f produces a "finite time average" $\frac{1}{L}\int_0^L f(T_t\gamma)dt$. If a large number of repeated measurements of f produce the same value we may say that the system is in "equilibrium with respect to f", and it is reasonable to identify the measured value with the infinite time average (40). $f^*(\gamma)$ may be called the "equilibrium value of f", and (11) provides a formula for this value which does not require a knowledge of the solution operator T_t.

Physical systems which may be described in terms of an initial value problem are always isolated, i.e. the influence of the physical environment on the system may be neglected. However, in many situations the system of interest* Γ' interacts with the environment to an extent which may not be ignored. In such cases, the solution of the initial value problem no longer describes the behavior of the system. However, it may be possible to consider a larger system, consisting of the system Γ' plus the environment, for which the initial value problem is appropriate, i.e. the combined system is isolated.

If T_t' is the solution operator for the system Γ' when isolated, we assume that we can define a complete space $(\Gamma', \mathcal{A}', m', T_t', y')$. Similarly, we assume that the environment is a physical system Γ'' whose isolated behavior is described by the solution operator T_t'', and that we can define

---

* The symbol Γ' will be used to stand both for the system of interest and for its state space.

a complete space $(\Gamma'', \mathcal{A}'', m'', T_t'', y'')$. We assume that y' and y'' both have dimension k. Let us suppose, for the moment, that there is no interaction among the systems. Let $\Gamma = \Gamma' \times \Gamma''$, $\mathcal{A} = \mathcal{A}' \times \mathcal{A}''$, $m = m' \times m''$, and let T_t be the natural transformation of Γ induced by T_t' and T_t'', i.e.

$$T_t\gamma = T_t(\gamma', \gamma'') = (T_t'\gamma', T_t''\gamma'') . \tag{41}$$

Then it is easy to show that $(\Gamma, \mathcal{A}, m, T_t)$ is a measure-preserving space and all the invariants y_j', y_j'', ; $j = 1, \ldots, k$; are invariants of T_t.

Now suppose there is interaction between Γ' and Γ''. Then T_t will no longer be given by (41), nor, in general, will T_t be measure-preserving with respect to $(\Gamma, \mathcal{A}, m)$, nor will all the y_j', y_j'' be invariants of T_t. However, if the magnitude of the physical interaction is small, we may suppose that some of the above features are preserved. This leads to the definition of "weak interaction" given in the previous section. We assume that $(\Gamma, \mathcal{A}, m, T_t, y)$ is a complete space where

$$y = \left(y_1, \ldots y_k\right) = y' + y'' . \tag{42}$$

Some of the requirements of the definition of weak interaction, in particular (42), may be met only in an approximate sense*.

One further physical condition must be met before we can apply our theory: Γ'' must be "large" compared to Γ'. By this we mean that the behavior of the system Γ'' is not appreciably affected by its interaction with Γ'. In particular, the value of $y = y'' + y'$ is not appreciably affected by fluctuations

---

* Thus, e.g., there may be some "interaction parameter", ϵ, and (42) may be an asymptotic relation valid for $\epsilon \to 0$.

in the value of y'. This condition may be viewed as the physical interpretation of (14), which, in the applications, can be expected to hold only in an asymptotic sense.

Now if $(\Gamma'', \mathcal{A}'', m'', T_t'', y'') \gg (\Gamma', \mathcal{A}', m', T_t', y')$ we may apply theorem II: If $f(\gamma')$ is some physically observable quantity associated with the system Γ' in the state γ', the "infinite time average"

$$f^*(\gamma) = \lim_{L\to\infty} \frac{1}{L} \int_0^L f(T_t\gamma)dt \tag{43}$$

exists a.e. with respect to m, and (29) provides a simple formula for f^* which does not require a knowledge of T_t: $f^* = \bar{f}(\alpha)$.

But presumably we wish to predict the value of f^* which we may expect to measure in a given situation. For this, we must know the value of α. The significance of the corollary to theorem II is that α can be obtained from a knowledge of $\bar{y}'$, which is the infinite time average of $y'(\gamma')$, and thus is presumably measurable. Knowing $\bar{y}'$ we can then predict that a measurement of f will produce the value $\bar{f}[\alpha(\bar{y}')]$.

Thus the situation is somewhat analogous to the case of the isolated system. There the value of f one would expect to measure was given by f^*, and f^* was uniquely determined by the (invariant) value of y. Here the value of f we expect to measure is given by $\bar{f}$, and $\bar{f}$ is uniquely determined by the value of $\bar{y}'$.

B. The Generalized Temperture

Let us suppose that the system of interest, Γ', itself consists of a fixed number, q, of systems in weak interaction. We suppose that q real physical systems are in weak interaction with one another, and the combined system is in weak physical interaction with its environment.

Mathematically, in addition to the hypothesis of Theorem II, we assume that $(\Gamma_j, \mathcal{A}_j, m_j, T_{jt}, y_{(j)})$ is a complete space for $j = 1, \ldots q$; where $y_{(j)}(\gamma_j) = \left\{y_{(j)1}, \ldots, y_{(j)k}\right\}$; $(\Gamma', \mathcal{A}', m') = \Gamma_1 \times \ldots \times \Gamma_q, \mathcal{A}_1 \times \ldots \times \mathcal{A}_q, m_1 \times \ldots \times m_q)$; and $y' = \sum_{j=1}^{q} y_{(j)}$.

We note that these assumptions imply that the complete spaces $(\Gamma_j, \mathcal{A}_j, m_j, T_{jt}, y_{(j)})$; $j=1,\ldots,q$ are in weak interaction* . We also note that $D' = \bigcap_{j=1}^{q} D_j$, where D_j is the set of vectors α in R^k such that

$$\int e^{-a \cdot y_{(j)}} dm_j < \infty .$$

From the conclusions of Theorem II, we have

$$(44) \qquad z'(\alpha) = \int e^{-\alpha \cdot y'} \, dm' = \prod_{j=1}^{q} z_j(\alpha); \; z_j(\alpha) = \int e^{-\alpha \cdot y_{(j)}} dm_j .$$

For functions $f(\gamma_1)$ such that $f(\gamma_1)e^{-\alpha \cdot y_{(1)}}$ is integrable on $(\Gamma_1, \mathcal{A}_1, m_1)$ for each α in D_1,

$$(45) \quad \bar{f}(\alpha) = [z'(\alpha)]^{-1} \int f(\gamma_1) e^{-\alpha \cdot y'(\gamma')} dm' = (z')^{-1} \int f e^{-\alpha \cdot y_{(1)}} dm_1 \cdot \prod_{j=2}^{q} z_j$$

$$= (z_1)^{-1} \int f(\gamma_1) e^{-\alpha \cdot y_{(1)}} dm_1 .$$

If we omit subscripts "1" associated with the first component of the system of interest, i.e. with the space Γ_1 , we have

$$(46) \qquad z(\alpha) = \int e^{-\alpha \cdot y} dm$$

---

*The definition of "weak interaction" for more than two complete spaces is an obvious extension of our earlier definition.

$$\bar{f}(\alpha) = [z(\alpha)]^{-1} \int f(\gamma) e^{-\alpha \cdot y(\gamma)} \, dm \,. \tag{47}$$

Note that although we have singled out the system Γ_1 for special treatment we could have done the same for any system Γ_j, $j=1,\ldots,q$; with similar results. If we also omit the accent ' in (23) and (29) we see that these equations are identical to (46) and (47). Now let us compare the two cases: (I) The system represented by Γ is in weak interaction with its environment. (II) The system represented by Γ is in weak interaction with q-1 other systems and the combined system is in weak interaction with its environment.

We conclude from the above remarks that $\bar{f}$ is the same in cases (I) and (II) provided only that α is the same . Furthermore, suppose we now measure the quantity $f_j(\gamma_j)$ in system Γ_j , for $j=1,\ldots,q$. We would expect in each case to obtain a value $\bar{f}_j(\alpha)$ where α is the same for every j, since we have seen that α is uniquely determined by $\bar{y}'$.

Up until now, we have viewed the parameter α merely as a mathematical convenience. In fact we went to great lengths to show that $\bar{f}$ could be expressed as a function of $\bar{y}'$ rather than as a function of α. However, we see now that α may be viewed as a parameter of the system which is the same for systems in weak interaction with one another. From this point of view, α takes on physical significance and its behavior is very much like a temperature. For this reason, we shall refer to α as the generalized temperature of the system.

C. The generalized "Second Law"

Let $\beta = (\beta_1,\ldots,\beta_r)$ be a vector in R^r, and let us assume that y' and T_t' depend parametrically on β. In addition, we assume that, for each fixed value of β in some region I_β of R^r, all the hypotheses of Theorem II hold; hence all the conclusions also hold. Then our main results are summarized in the formulas (we omit the accent)

$$(48) \qquad z = z(\alpha;\beta) = \int e^{-\alpha\cdot y(\gamma;\beta)}dm$$

$$(49) \qquad \bar{f} = \bar{f}(\alpha;\beta) = z^{-1}\int f(\gamma)e^{-\alpha\cdot y(\gamma;\beta)}dm.$$

If we assume that we can differentiate formally, we have

$$(50) \quad d(\log z) = -z^{-1}\int e^{-\alpha\cdot y}\left\{\sum_i y_i d\alpha_i + \sum_i\left[\sum_k \alpha_k \frac{\partial y_k}{\partial\beta_i}\right]d\beta_i\right\}dm$$

$$(51) \quad d(\log z) = -\sum_i \bar{y}_i d\alpha_i - \sum_i\left[\sum_k \alpha_k \overline{\left(\frac{\partial y_k}{\partial\beta_i}\right)}\right]d\beta_i .$$

Let us define

$$(52) \quad q = q(\alpha,\beta,\gamma) = z^{-1}e^{-\alpha\cdot y}\,;\; H = H(\alpha,\beta) = \int q\log q\,dm.$$

Then $\int q\,dm = 1$ and $\log q + \log z + \alpha\cdot y = 0$. If we multiply this equation by q and integrate with respect to m over Γ we obtain

$$(53) \qquad H + \log z + \alpha\cdot\bar{y} = 0 .$$

Differentiating (53) we have $dH + d(\log z) + \sum_i \alpha_i d(\bar{y}_i) + \sum_i \bar{y}_i d\alpha_i = 0$. Combining this with (51) yields

$$(54) \quad -dH = \sum_i \alpha_i d(\bar{y}_i) - \sum_i\left[\sum_k \alpha_k \overline{\left(\frac{\partial y_k}{\partial\beta_i}\right)}\right]d\beta_i .$$

This equation will be called the <u>generalized second law</u> because of its similarity to the second law of Thermodynamics. Indeed we shall compare it directly with that law shortly. For later reference we shall also need the following equations which are obtained from (48) by differentiation:

$$(55) \qquad \bar{y}_k = -\frac{\partial}{\partial\alpha_k}\log z ,$$

$$(56) \qquad \sum_k \alpha_k \overline{\left(\frac{\partial y_k}{\partial\beta_i}\right)} = -\frac{\partial}{\partial\beta_i}\log z.$$

D. Classical mechanics

A mechanical system with s degrees of freedom may be described in terms of generalized coordinates $q_1,\ldots,q_s$ and conjugate momenta $p_1,\ldots,p_s$ which are solutions of Hamilton's equations

$$\frac{dq_i}{dt} = \frac{\partial E}{\partial p_i} \quad , \quad \frac{dp_i}{dt} = -\frac{dE}{\partial q_i} \quad ; \quad i = 1,\ldots,s. \tag{57}$$

Here $E = E(q_1,\ldots,q_s,p_1,\ldots,p_s,t)$ is the total energy or "Hamiltonian" of the system. We shall denote the 2s coordinates and momenta by a single vector $\rho = \{\rho_1,\ldots,\rho_{2s}\} = \{q_1,\ldots,q_s,p_1,\ldots,p_s\}$ which may be thought of as a point in 2s dimensional euclidean space or "phase space", P. We shall assume that the mechanical system is "conservative", i.e. E is independent of t, and that E depends on r parameters $\beta_1,\ldots,\beta_r$. Thus $E = E(\rho;\beta)$ where $\beta = \{\beta_1,\ldots,\beta_r\}$, and β varies in some region I_β of R^r. We assume that as a function of (ρ,β) defined in $P \times I_\beta$, E has the necessary properties such that the initial value problem defined by (57) and

$$\rho = \rho^o \quad \text{at } t = 0 \tag{58}$$

has a unique solution $\rho(t;\rho^o,\beta)$ which has continuous first derivatives with respect to all its arguments in $R^1 \times P \times I_\beta$. (Here R^1 is the set $-\infty < t < \infty$.) If we define the solution operator T_t by

$$T_t\rho^o = \rho(t;\rho^o,\beta) \; ; \; -\infty < t < \infty \; , \; \rho^o \in P \; , \; \beta \in I_\beta, \tag{59}$$

then for each fixed t and β, T_t is a continuously differentiable transformation of phase space into itself.

Now for any real s, the two functions $\rho(t;T_s\rho^o,\beta)$ and $\rho(t+s;\rho^o,\beta)$ are both equal to $T_s\rho^o$ at T = 0. It follows from the uniqueness assertion that the two functions are equal and hence that

(60) $T_{t+s} = T_t\, T_s$ for all real t and s.

In particular $T_{-t}\, T_t = T_o = I$, where I is the identity transformation.

Thus for every t, T_t has an inverse and $T_t^{-1} = T_{-t}$.

It follows easily from (57) that

(61) $$\frac{d}{dt} E[q_i(t),p_i(t);\beta] = \sum_i \left[\frac{\partial E}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial E}{\partial p_i}\frac{dp_i}{dt}\right] = \sum_i \left[\frac{\partial E}{\partial q_i}\frac{\partial E}{\partial p_i} - \frac{\partial E}{\partial p_i}\frac{\partial E}{\partial q_i}\right] = 0 .$$

Thus $E[\rho(t);\beta] = E[\rho(0);\beta]$, or

(62) $E[T_t\rho^o;\beta] = E[\rho^o;\beta]$ for all $t \in R^1$ and $\rho^o \in \Gamma$.

The following Lemma contains assertions which are needed in order to apply our theory to mechanical systems. The proof appears in Appendix 1.

Lemma VI:

Let μ denote Lebesgue measure, and $\mathcal{L}$ denote the class of Lebesgue measurable sets in phase space Γ . Let $\mathcal{M}$ denote the class of Lebesque measurable sets in R^1. Then

(I) The transformations T_t are measure-preserving with respect to $(\Gamma,\mathcal{L},\mu)$.

(II) $T_t\rho$ is a measurable function of t and ρ, i.e. $T_t\rho$ is a measurable transformation from $(R^1 \times \Gamma, \mathcal{M}\times\mathcal{L})$ into $(\Gamma,\mathcal{L})$.

(III) E is a measurable function with respect to $(\Gamma,\mathcal{L},\mu)$.

From (III) and (62) it follows that E is an invariant function for each fixed β, and from (I), (II) and (60) that $(\Gamma,\mathcal{L},\mu,T_t)$ is a measure-preserving space.

We now restrict our considerations to mechanical systems for which the function E is, by itself, a complete set of invariants* . Experimental evidence indicates that this assertion leads to consistent results for a surprisingly large class of systems. Thus $(\Gamma,\mathcal{L},\mu,T_t,E)$ is a complete space.

---

* For a discussion of certain mechanical systems where E alone is not a complete set of invariants, see [2] .

For many Hamiltonians of practical interest, it can be verified that for every finite interval, I, in R^1,

$$\mu\left\{E^{-1}(I)\right\} = \mu\left\{\rho \ni E(\rho) \in I\right\} < \infty. \tag{63}$$

We restrict consideration to mechanical systems satisfying this condition, also.

For isolated mechanical systems satisfying the above conditions, we may now apply Theorem I. If $f(\rho)$ is a "phase function", i.e. if f is integrable on $(P,\mathcal{L},\mu)$, then the infinite time average $f^*(\rho)$ exists a.e. If the system is in equilibrium with respect to f, we may predict that a measurement of f will produce the value f^*, and we have the following formula:

$$f^*\left[E^{-1}(x)\right] = \lim_{j\to\infty} \frac{\int_{\left\{\rho \ni c^j < E(\rho;\beta) - x \le d^j\right\}} f(\rho)\,d\mu}{\mu\left\{\rho \ni c^j < E(\rho;\beta) - x \le d^j\right\}}. \tag{64}$$

$f^*\left[E^{-1}(x)\right]$ denotes the value of $f^*(\rho)$ common to all ρ for which $E(\rho)=x$; c^j, d^j are sequences of numbers, which depend on x, for which $c^j < 0 \le d^j$, $\lim_{j\to\infty}(d^j - c^j) = 0$. Formula (64) is sometimes referred to as the microcanonical distribution.

If our mechanical system interacts physically with its environment, we assume that we may treat the system and environment together as a mechanical system whose states are points in a phase space $P = P' \times P''$; where P' is the phase space of the system of interest and P'' is the phase space of the environment. It is certainly true that the transformations T_t of P determined by Hamilton's equations and the total energy E are measure-preserving with respect to $(P,\mathcal{L},\mu)$, where u is Lebesque measure in P, and that E is an invariant of T_t. We again restrict consideration to systems for which E is

a complete set of invariants. If the magnitude of the physical interaction is small then E will be approximately equal to the sum of the energies of the component systems. Thus the condition

$$E = E' + E'' \tag{65}$$

will be satisfied in an asymptotic sense. It is easy to show that $(P, \mathcal{L}, \mu) = (P' \times P'', \mathcal{L}' \times \mathcal{L}'', \mu' \times \mu'')$. Thus we may assume that the complete spaces $(P', \mathcal{L}', \mu', T_t', E')$, $(P'', \mathcal{L}'', \mu'', T_t'', E'')$ are in weak interaction, according to our definition of that term. For most Hamiltonians E' of practical interest, the set D' of all real numbers α such that

$$z'(\alpha) = \int_{P'} e^{-\alpha E'(\rho')} d\mu' < \infty \quad , \tag{66}$$

consists of the set $\{0 < \alpha < \infty\}$, which is an open connected set. Thus, in order to apply the conclusions of Theorem II and its corallary, we need only assume that the environment is "large" in the sense defined.

Let us now examine the consequences of Theorem II. To conform with general usage we shall denote α by θ, and shall omit the accent. For θ in $D = \{0 < \theta < \infty\}$,

$$z(\theta, \beta) = \int_P e^{-\theta E(\rho, \beta)} d\mu \ . \tag{67}$$

If $f(\rho)e^{-\theta E(\rho,\beta)}$ is integrable on $(P, \mathcal{L}, \mu)$ for each θ in D, then

$$\bar{f}(\theta, \beta) = z^{-1} \int_P f(\rho) e^{-\theta E(\rho, \beta)} d\mu \ . \tag{68}$$

Formulas (67) and (68) are sometimes referred to as the <u>canonical distribution</u>.

As in Section 2C, we define

(69) $$q = z^{-1} e^{-\theta E} ,$$

and

(70) $$H = \int q \log q \, d\mu .$$

Then from (53)

(71) $$H + \log z + \theta \bar{E} = 0 .$$

From (54)

(72) $$- dH = \theta d(\bar{E}) - \theta \sum_i \overline{\left(\frac{\partial E}{\partial \beta_i} \right)} \, d\beta_i .$$

Systems of the type we have been discussing are studied in thermodynamics. The main result of this study is the "second law of thermodynamics" which may be written as

(73) $$TdS = dU + \sum_i \mathcal{P}_i \, d\beta_i$$

where $\mathcal{P}_i$ denotes the "generalized force" corresponding to the "external variable" β_i, T denotes absolute temperature, U internal energy, and S entropy. Comparing (72) and (73) we see that these equations are identical if we set

(74) $$\frac{1}{\theta} = kT; \; -kH = S; \; \bar{E} = U; \; - \overline{\left(\frac{\partial E}{\partial \beta_i} \right)} = \mathcal{P}_i .$$

The constant k can have any real value. By convention it is fixed at a certain value and this determines a convenient temperature scale. In terms of the identities (74), z, which is called the <u>partition function</u>, may be written

(75) $$z = \int_P e^{-\frac{1}{kT} E(q_i, p_i; \beta)} \, dq_i dp_i .$$

From (71) we have

(76) $$-kT \log z = U - TS .$$

In thermodynamics the quantity U-TS is called the "free energy" and is denoted by A. Thus

$$A = - kT\log z = - \frac{1}{\beta} \log z \quad . \tag{77}$$

From (55) we obtain

$$U = kT^2 \frac{\partial}{\partial T} \log z = - T^2 \frac{\partial}{\partial T} \left(\frac{A}{T}\right) \quad , \tag{78}$$

and from (56)

$$\mathcal{P}_i = kT \frac{\partial}{\partial \beta_i} \log z = - \frac{\partial A}{\partial \beta_i} \quad . \tag{79}$$

Equations (76), (77), (78), and (79) express the thermodynamic variables $\mathcal{P}_i$, U, A, and S as functions of T and β which involve only the partition function and its logarithmic derivatives.

E. The grand canonical distribution

We now consider a mechanical system consisting of δ different kinds of particles. If n_i denotes the number of particles of the ith kind, and s_i the number of degrees of freedom of the ith kind of particle, then the states of the system can be represented in a phase space with points $\{q_{ijk}, p_{ijk}\}$ where $i = 1,\ldots,\delta$ labels the kind of particle; $j = 1,\ldots,n_i$ numbers the particles of that kind; and $k = 1,\ldots,s_i$ determines the coordinates corresponding to each degree of freedom. The vector $n = \{n_1,\ldots,n_\delta\}$ determines the "composition" of the system, i.e. the number of particles of each kind.

For a given composition, n, we have a phase space P_n of points $\rho_n = (q_{ijk}, p_{ijk})$. We assume that as a function of ρ_n, β defined in $P_n \times I_\beta$, the Hamiltonian $E_n(\rho_n, \beta)$ has the necessary properties such that the initial value problem has a unique continuously differentiable solution. If μ_n denotes Lebesgue measure, $\mathcal{L}_n$ denotes the class of Lebesgue measurable sets in P_n, and if S_{nt} denotes the solution operator for the system, then we know from the discussion of the preceding section that $(P_n, \mathcal{L}_n, \mu_n, S_{nt})$ is a measure-preserving space and E_n is an invariant function. As before we shall restrict consideration to systems for which E_n is, by itself, a complete

set of invariants. Thus $(P_n, L_n, \mu_n, S_{nt}, E_n)$ is a complete space.

As in the preceding section, we shall assume that our physical system interacts with its environment, but the interaction will include not only the energy, but the particles themselves. Thus we shall suppose that particles can be exchanged between the system and its environment. In such a situation it is no longer possible to represent the state of the system by a point in a single phase space P_n, but rather as a point in $\bigcup_n P_n$. In this connection, the following Theorem will be used. The proof appears in Appendix 2.

Theorem III .

Let $\mathcal{N}$ be the space of vectors $n = \{n_1, \ldots, n_\delta\}$ where n_i is a non-negative integer for $i = 1, \ldots, \delta$. For each $n \in \mathcal{N}$, let $(P_n, \mathcal{L}_n, \mu_n, S_{nt}, g_{(n)})$ be a complete space, where $g_{(n)} = g_{(n)1}, \ldots, g_{(n)k} = G_{(n)}(\rho_n)$, and ρ_n denotes a point in P_n. Let

$$\Gamma = \bigcup_{n \in \mathcal{N}} P_n \quad . \tag{80}$$

Let $\mathcal{A}$ be the class of subsets A of Γ such that

$$A \cap P_n \in \mathcal{L}_n \qquad \text{for every } n \in \mathcal{N} \; . \tag{81}$$

For $A \in \mathcal{A}$, let

$$m(A) = \sum_{n \in \mathcal{N}} \mu_n(A \cap P_n) \; . \tag{82}$$

For each $\gamma = \rho_n$ in Γ, let

$$T_t \gamma = S_{nt} \rho_n \quad . \tag{83}$$

Let $n(\gamma) = [n_1, \ldots, n_\delta]$ be the vector function defined by

(84) $$n(\gamma) = n(\rho_n) = n ,$$

and let

(85) $$g(\gamma) = g(\rho_n) = g_{(n)}(\rho_n) .$$

Let $y(\gamma)$ be the $(\delta+k)$ - component vector function defined by

(86) $$y(\gamma) = \{n(\gamma), g(\gamma)\}.$$

Then $(\Gamma, \mathcal{A}, m, T_t, y)$ is a complete space. $f(\gamma) = f_n(\rho_n)$ is a measurable function with respect to $(\Gamma, \mathcal{A})$ if and only if $f_n(\rho_n)$ is a measurable function with respect to $(P_n, \mathcal{L}_n)$ for every $n \in \mathcal{N}$. If $f(\gamma)$ is integrable on $(\Gamma, \mathcal{A}, m)$ then $f_n(\rho_n) = f(\rho_n)$ is integrable on $(P_n, \mathcal{L}_n, \mu_n)$ for each $n \in \mathcal{N}$,

(87) $$\int_\Gamma f(\gamma)dm = \sum_{n\in\mathcal{N}} \int_{P_n} f_n(\rho_n)d\mu_n ,$$

and the series converges absolutely. Conversely if $f_n(\rho_n)$ is non-negative and integrable on $(P_n, \mathcal{L}_n, \mu_n)$ for each $n \in \mathcal{N}$, and if the above series converges, then $f(\gamma) = f_n(\rho_n)$ in integrable on $(\Gamma, \mathcal{A}, m)$ and (87) holds.

We now apply this theorem setting $g_{(n)}(\rho_n) = E_n(\rho_n)$ (thus $k = 1$) . In order to apply theorem II we set $(\Gamma', \mathcal{A}', m', T_t', y') = (\Gamma, \mathcal{A}, m, T_t, y)$, and assume that the environment can be represented by a similar space $(\Gamma'', \mathcal{A}'', m'', T_t'', y'')$ where $(\Gamma'', \mathcal{A}'', m'', T_t'', y'') \gg (\Gamma', \mathcal{A}', m', T_t', y')$. The vector α will be denoted by $\{\lambda, \theta\}$ where $\lambda = \{\lambda_1, \ldots, \lambda_\delta\}$ and $\alpha \cdot y = \lambda \cdot n + \theta E_n$. In stating the results, we omit the accent: For $\{\lambda, \theta\}$ in some subset D of $R^{\delta+1}$ (which is assumed to be a non-empty, open, connected set)

(88) $$z(\lambda, \theta, \beta) = \int e^{-\alpha \cdot y} dm = \sum_{n\in\mathcal{N}} e^{-\lambda \cdot n} \int_{P_n} e^{-\theta E_n(\rho_n, \beta)} d\mu_n .$$

If $f(\gamma)e^{-\alpha \cdot y(\gamma)} = f_n(\rho_n)e^{-\lambda \cdot n - \theta E_n(\rho_n)}$ is integrable on $(\Gamma, \mathcal{A}, m)$ for every $\alpha = \{\lambda, \theta\}$ in D, then

$$(89)\qquad \bar{f}(\lambda,\theta,\beta) = z^{-1}\int fe^{-\alpha\cdot y}dm = z^{-1}\sum_{n\in\mathcal{N}} e^{-\lambda\cdot n}\int_{P_n} f_n(\rho_n)e^{-\theta E_n(\rho_n)}d\mu_n \ .$$

Formulas (88) and (89) are sometimes referred to as the Grand Canonical Distribution.

As in Section 2C we define

$$(90)\qquad q = z^{-1}e^{-\lambda\cdot n-\theta E_n} \ ,$$

$$(91)\qquad H = \int q\log q dm \ .$$

Then from (53)

$$(92)\qquad H + \log z + \lambda\cdot\bar{n} + \theta\bar{E}_n = 0 \ .$$

From (54)

$$(93)\qquad -dH = \theta d\bar{E}_n - \theta\sum_i \overline{\left(\frac{\partial E_n}{\partial\beta_i}\right)}\ d\beta_i + \sum_i \lambda_i d\bar{n}_i \ .$$

Systems of the type which we have been describing are also studied in thermodynamics. For this type of system the second law may be written

$$(94)\qquad TdS = dU + \sum_i \mathcal{P}_i d\beta_i - \sum_i \mu_i d\nu_i \ .$$

μ_i is the "partial free energy" or "partial thermal potential" associated with the i^{th} kind of particle, and ν_i is the average number of particles of the i^{th} kind. Comparing (93) and (94) we see that these equations are identical if we set

$$(95)\qquad \frac{1}{\theta} = kT \ ; \ -kH = S \ ; \ \bar{E}_n = U \ ; \ -\overline{\left(\frac{\partial E_n}{\partial\beta_i}\right)} = \mathcal{P}_i \ ; \ \frac{\lambda_i}{\theta} = -\mu_i \ ; \ \bar{n}_i = \nu_i \ .$$

In terms of these identities, z, which is called the grand partition function may be written

$$(96)\qquad z = \sum_{n_1,\ldots,n_\delta=0}^{\infty} \exp\left\{\sum_i \frac{\mu_i n_i}{kT}\right\}\int_{P_n} \exp\left\{-\frac{1}{kT}E_n(q_{ijk},p_{ijk};\beta)\right\}dq_{ijk}dp_{ijk} \ .$$

From (92) we have

$$-kT\log z = U - TS - \sum_i \mu_i \nu_i \quad . \tag{97}$$

From (55) we obtain

$$U = kT^2 \frac{\partial}{\partial T} \log z \tag{98}$$

and

$$\nu_i = kT \frac{\partial}{\partial \mu_i} \log z \quad , \tag{99}$$

and from (56)

$$\mathcal{P}_i = kT \frac{\partial}{\partial \beta_i} \log z \quad . \tag{100}$$

Equations (97), (98), (99), and (100) express the thermodynamic variables $\mathcal{P}_i$, ν_i, U, S as functions of T, μ_i, and β which involve only the grand partition function and its logarithmic derivatives.

Appendix 1 : Proof of Lemma VI.

(I) The transformations T_t are measure-preserving with respect to $(P, \mathcal{L}, \mu)$.

To prove (I) it must be shown that for every $L \in \mathcal{L}$ and for every $T_t^{-1} L \in \mathcal{L}$ and $\mu(T_t^{-1}L) = \mu(L)$. The proof depends on a famous theorem of of Liouville*:

A. For every fixed t, the Jacobian of the transformation T_t is everywhere equal to 1.

B. We shall use the following definition of Lebesque measure**:

Let C be the class consisting of the empty set $\emptyset$ and the open intervals $\{a < \rho < b\} = \{\rho \ni a_i < \rho_i < b_i;\ i = 1,\ldots,2s\}$ in P. For any set $G \in C$, let $\tau(G)$ denote the 2s-dimensional volume of G. For any subset A of P, set

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \tau(G_n) \ni G_n \in C;\ \bigcup_{n=1}^{\infty} G_n \supset A \right\} . \tag{A1.1}$$

---

*See [1]

**See [6] p.93.

μ^* is a set function defined on the class of all subsets of P and is called Lebesque outer measure. By definition, $\mathcal{L}$ is the class of all sets L such that

(A1.2) $\quad \mu^*(A) = \mu^*(A \cap L) + \mu^*(A - L)$ for every $A \subset P$;

and μ is the restriction of μ^* to the sets of $\mathcal{L}$.

C. Let $\mathcal{D}$ be the class of all continuously differentiable transformations of P with Jacobian everywhere equal to 1. Let C_1 be the class of all subsets E_1 of P such that $E_1 = TE$ for some $T \in \mathcal{D}$ and some $E \in C$. For every set $E_1 \in C_1$ let $\tau(E_1)$ be the 2s-dimensional volume of E_1. (Then $\tau(E_1) = \tau(E)$.) For any subset A of P, set

$$(A1.3) \qquad \mu_1^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \tau(E_n) \ni E_n \in C_1;\ \bigcup_{n=1}^{\infty} E_n \supset A \right\}.$$

We shall show that $\mu_1^*(A) \equiv \mu^*(A)$. In so doing we shall make use of the following theorem*:

D. Every open set in P is the union of a countable class of disjoint, half-open intervals.

E. For every set E_1 in C_1, and every $\epsilon > 0$, there exists a sequence of sets G^k such that $G^k \in C$, $\bigcup_k G^k \supset E_1$, and $\sum_k \tau(G^k) \leq \tau(E_1) + \epsilon$:

If F is any half-open interval $(a,b]$, let $\tau(F)$ denote the 2s-dimensional volumne of F, i.e. $\tau(F) = \prod_{i=1}^{2s} (b_i - a_i)$. Given $\delta > 0$, there exists an open interval $G = (a,c)$ such that $F \subset G$ and $\tau(G) = \prod_{i=1}^{2s} (c_i - a_i) \leq \tau(F) + \delta$.

By D, there exists a sequence of disjoint, half-open intervals F^k whose union is E_1. For each F^k choose an open interval G^k such that $F^k \subset G^k$ and $\tau(G^k) \leq \tau(F^k) + \frac{\epsilon}{2^k}$. Then $\bigcup_k G^k \supset \bigcup_k F^k = E_1$, and $\sum_k \tau(G^k) \leq \epsilon + \sum_k \tau(F^k) = \epsilon + \tau(E_1)$.

---

*See [6], p 126.

F. $\mu_1^*(A) = \mu^*(A)$ for every $A \subset P$:

Since $C \subset C_1$ it follows that for every $A \subset P$

(A1.4) $$\mu_1^*(A) \leq \mu^*(A) .$$

Given any sequence of sets $E_n \in C_1$ such that $\bigcup_n E_n \supset A$, and given any $\epsilon > 0$, we may, by E , choose sets $G_n^k \in C$ such that for each n $\bigcup_{k=1}^{\infty} G_n^k \supset E_n$ and $\sum_k \tau(G_n^k) \leq \tau(E_n) + \frac{\epsilon}{2n}$. Then $\bigcup_{n,k} G_n^k \supset A$ and $\sum_{n,k} \tau(G_n^k) \leq \sum_n \left[\tau(E_n) + \frac{\epsilon}{2n}\right] \leq \epsilon + \sum_n \tau(E_n)$. If we renumber the G_n^k with a single index we have $\bigcup_n G_n \supset A$ and $\sum_n \tau(G_n) \leq \epsilon + \sum_n \tau(E_n)$. Since ϵ was arbitrary it follows that

(A1.5) $$\mu_1^*(A) \geq \mu^*(A) .$$

G, For every $L \in \mathcal{L}$ and every t, $T_t^{-1}L \in \mathcal{L}$ and $\mu(T_t^{-1}L) = \mu(L)$:

For any $A \subset P$

(A1.6) $$\mu^*(A) = \mu_1^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \tau(E_n) \ni E_n \in C_1;\ \bigcup_{n=1}^{\infty} E_n \supset A \right\}$$

$$= \inf \left\{ \sum_{n=1}^{\infty} \tau(T_t^{-1}E_n) \ni T_t^{-1}E_n \in C_1;\ \bigcup_{n=1}^{\infty} T_t^{-1}E_n \supset T_t^{-1}A \right\} = \mu_1^*(T_t^{-1}A)$$

$$= \mu^*(T_t^{-1}A).$$

Let L be any set in $\mathcal{L}$. For any set $B \subset P$ let $A = T_t B$. Then $T_t^{-1}A = B$ and

(A1.7) $$\mu^*(B) = \mu^*(T_t^{-1}A) = \mu^*(A) = \mu^*(A \cap L) + \mu^*(A - L)$$

$$= \mu^*\left[T_t^{-1}(A \cap L)\right] + \mu^*\left[T_t^{-1}(A - L)\right]$$

$$= \mu^*\left[T_t^{-1}A \cap T_t^{-1}L\right] + \mu^*\left[T_t^{-1}A - T_t^{-1}L\right]$$

$$= \mu^*(B \cap T_t^{-1}L) + \mu^*(B - T_t^{-1}L).$$

Hence $T_t^{-1}L \epsilon \mathcal{L}$ and

(A1.8) $\mu(T_t^{-1}L) = \mu^*(T_t^{-1}L) = \mu^*(L) = \mu(L)$.

II. $T_t\rho$ is a measurable function of t and ρ, i.e. $T_t\rho$ is a measurable transformation from $(R^1 \times P, \mathcal{M} \times \mathcal{L})$ into $(P, \mathcal{L})$:

Given a measure space $(\Gamma, \mathcal{A}, m)$, let Z be the class of all subsets of sets of measure zero. Let $\bar{\mathcal{A}}$ be the class of sets of the form zUA, where $z \epsilon Z$ and $A \epsilon \mathcal{A}$. Then $\bar{\mathcal{A}}$ is a completely additive class of sets containing $\mathcal{A}$ and the extension $\bar{m}$ of m to $\bar{\mathcal{A}}$ defined by

(A1.9) $\bar{m}(zUA) = m(A)$

is a measure. $\bar{m}$ is called the completion of m.

Let ν denote Lebesque measure on $(R^1, \mathcal{M})$. It can be shown that the Borel sets in R^k are a subclass of the Lebesque measurable sets, and the restriction of Lebesque measure to the Borel sets is a measure whose completion is Lebesque measure. Thus if $\hat{\mathcal{M}}$ denotes the Borel sets in R^1 and $\hat{\nu}$ denotes the restriction of ν to $\hat{\mathcal{M}}$, then $\bar{\hat{\mathcal{M}}} = \mathcal{M}$ and $\bar{\hat{\nu}} = \nu$. We shall also denote the Borel sets in P by $\hat{\mathcal{L}}$ and the restriction of μ to $\hat{\mathcal{L}}$ by $\hat{\mu}$. As before $\bar{\hat{\mathcal{L}}} = \mathcal{L}$ and $\bar{\hat{\mu}} = \mu$.

It can easily be shown that if f is a measurable transformation from $(X, \mathcal{A}, \sigma)$ into $(Y, \mathcal{B}, \tau)$ such that $\sigma[f^{-1}(B)] = 0$ whenever $\tau(B) = 0$, then f is a measurable transformation from $(X, \bar{\mathcal{A}}, \bar{\sigma})$ into $(Y, \bar{\mathcal{B}}, \bar{\tau})$. Now $f(t,\rho) = T_t\rho$ is a continuous transformation from $R^1 \times P$ into P. From this it can be shown that f is measurable from $(R_1 \times P, \hat{\mathcal{M}} \times \hat{\mathcal{L}}, \hat{\nu} \times \hat{\mu})$ into $(P, \hat{\mathcal{L}}, \hat{\mu})$. Thus it will follow that f is measurable from $(R_1 \times P, \mathcal{M} \times \mathcal{L}, \nu \times \mu)$ into $(P, \mathcal{L}, \mu)$ if we can show that

(A1.10) $\hat{\nu} \times \hat{\mu}\,[f^{-1}(L)] = 0$ whenever $\hat{\mu}[L] = 0$.

Now if G is any subset of $R_1 \times P$, the set $G_t = \{\rho \ni (t,\rho)\epsilon G\}$ is a subset of P called the section of G determined by t. We shall show that every section $[f^{-1}(L)]_t$ has $\hat{\mu}$-measure zero. From this it follows* that $\hat{\nu} \times \hat{\mu}[f^{-1}(L)] = 0$. Let $f_t(\rho) = f(t,\rho) = T_t\rho$. We have seen that for each fixed t, f_t is a continuously differentiable transformation of P into itself with Jacobian 1. Now

$$(A1.11) \qquad [f^{-1}(L)]_t = \left\{\rho \ni f(t,\rho)\epsilon L\right\} = \left\{\rho \ni f_t(\rho)\epsilon L\right\} = f_t^{-1}(L) .$$

If L is any Borel set of measure zero and ϵ is any positive number, there exists an open set $\mathcal{O}$ of volume ϵ containing L. Then $f_t^{-1}(\mathcal{O})$ is also of volume ϵ and contains $f_t^{-1}(L)$. Since ϵ was arbitrary, it follows that $\hat{\mu}\{f_t^{-1}(L)\} = \hat{\mu}\{[f^{-1}(L)]_t\} = 0$, and the proof is complete.

III. E is a measurable function with respect to $(P, \mathcal{L}, \mu)$.

This is a simple consequence of the fact that E is a continuous function on P (See [6], p.146).

Appendix II. Proof of Theorem III.**

1. $\mathcal{A}$ is a completely additive class of subsets of Γ.

$\phi\epsilon\mathcal{A}$. Let $A\epsilon\mathcal{A}$; then $(\Gamma - A)\cap P_n = P_n - A\cap P_n \epsilon \mathcal{L}_n$ for every $n\epsilon\mathcal{N}$; therefore, $(\Gamma - A)\epsilon\mathcal{A}$. Let $A_1, A_2, \ldots$ be in $\mathcal{A}$ and set $A = \bigcup_{k=1}^{\infty} A_k$. Then $A\cap P_n = (\bigcup_k A_k \cap P_n)\epsilon \mathcal{L}_n$ for every $n\epsilon\mathcal{N}$; therefore, $A\epsilon\mathcal{A}$.

2. m is a σ - finite measure function defined on $\mathcal{A}$:

Let $A_1, A_2, \ldots$ be a sequence of disjoint sets in $\mathcal{A}$. Then $m(\bigcup_k A_k)$ $= \sum_{n\epsilon\mathcal{N}} \mu_n(\bigcup_k A_k \cap P_n) = \sum_n \sum_k \mu_n(A_k \cap P_n) = \sum_k \sum_n \mu_n(A_k \cap P_n) = \sum_k m(A_k)$.

---

* See, e.g. [3], Sec. 36, Theorem A.

** The definition of the integral and other measure-theoretic concepts used here are based on [6].

Let $P_n = \bigcup_{k=1}^{\infty} L_{nk}$; $\mu_n(L_{nk}) < \infty$. Then $\Gamma = \bigcup_n \bigcup_k L_{nk}$, and $m(L_{nk}) = \mu_n(L_{nk}) < \infty$; hence m is σ - finite.

3. $f(\gamma) = f_n(\rho_n)$ is a measurable function with respect to $(\Gamma, \mathcal{A})$ if and only if $f_n(\rho_n)$ is a measurable function with respect to $(P_n, \mathcal{L}_n)$ for every $n \in \mathcal{N}$:

For any Borel set $B \subset R^1$, $f^{-1}(B) \cap P_n = \{\gamma \ni f(\gamma) \in B\} \cap P_n = \{\rho_n \ni f_n(\rho_n) \in B\}$ $= f_n^{-1}(B)$. Hence $f^{-1}(B) \in \mathcal{A}$ if and only if $f_n^{-1}(B) \in \mathcal{L}_n$ for every $n \in \mathcal{N}$.

4. Let $f(\gamma) = \sum_{k=1}^{p} a_k C_{A_k}(\gamma)$ be a simple function.* Then $f_n(\rho_n) = f(\rho_n)$ is simple for each $n \in \mathcal{N}$. If $f(\gamma)$ is integrable on $(\Gamma, \mathcal{A}, m)$ then $f_n(\rho_n)$ is integrable on $(P_n, \mathcal{L}_n, \mu_n)$ for each $n \in \mathcal{N}$,

$$\text{(A2.1)} \qquad \int_{\Gamma} f(\gamma) dm = \sum_{n \in \mathcal{N}} \int_{P_n} f_n(\rho_n) d\mu_n$$

and the series converges. Conversely, if $f_n(\rho_n)$ is integrable on $(P_n, \mathcal{L}_n, \mu_n)$ for each $n \in \mathcal{N}$ and the series converges, then $f(\gamma)$ is integrable on $(\Gamma, \mathcal{A}, m)$:

$$\text{(A2.2)} \qquad \int_{\Gamma} f(\gamma) dm = \sum_k a_k m(A_k) = \sum_k a_k \sum_n \mu_n(A_k \cap P_n) = \sum_k a_k \sum_n \int_{P_n} C_{A_k}(\rho_n) d\mu_n$$

$$= \sum_n \int_{P_n} f_n(\rho_n) d\mu_n \ .$$

Assertion 4 now follows from (A2.2) and the definition of integrability for simple functions.

5. Let $f(\gamma)$ be non-negative and measurable. Then $f_n(\rho_n) = f(\rho_n)$ is non-negative and measurable for each $n \in \mathcal{N}$. If $f(\gamma)$ is integrable then $f_n(\rho_n)$ is integrable for each $n \in \mathcal{N}$, (A2.1) is valid, and the series converges:

---

* $C_{A_k}(\gamma)$ denotes the characteristic function of the set A_k. For the definition of a simple function see [6].

If f is integrable, then, by definition, there exists a non-decreasing sequence $f_1, f_2, \ldots$ of non-negative simple functions, each integrable on $(\Gamma, \mathcal{A}, m)$ such that for each $\gamma \in \Gamma$ $\lim_{k\to\infty} f_k(\gamma) = f(\gamma)$, and such that $\lim_{k\to\infty} \int_\Gamma f_k dm < \infty$. By definition

$$\text{(A2.3)} \qquad \int_\Gamma f(\gamma)dm = \lim_{k\to\infty} \int_\Gamma f_k(\gamma)dm \ .$$

Set $C_{km} = \int_{P_n} f_k(\rho_n)\, d\mu_n$. Then $C_{k+1,n} \geq C_{k,n}$. In the following argument we use the fact that a convergent series of non-negative terms may be rearranged at will.

$$\text{(A2.4)} \quad \int_\Gamma f(\gamma)dm = \lim_{k\to\infty} \sum_{n\in\mathcal{N}} C_{kn} = \sum_{k=1}^{\infty} \left[\sum_{n\in\mathcal{N}} C_{k+1,n} - \sum_{n\in\mathcal{N}} C_{k,n} \right]$$

$$= \sum_k \sum_n (C_{k+1,n} - C_{k,n}) = \sum_n \sum_k (C_{k+1,n} - C_{k,n})$$

$$= \sum_n \lim_{k\to\infty} C_{k,n} = \sum_n \lim_{k\to\infty} \int_{P_n} f_k(\rho_n) d\mu_n$$

$$= \sum_{n\in\mathcal{N}} \int_{P_n} f(\rho_n) d\mu_n = \sum_{n\in\mathcal{N}} \int_{P_n} f_n(\rho_n) d\mu_n \quad .$$

6. Let $f_n(\rho_n)$ be non-negative and measurable for each $n\in\mathcal{N}$. Then $f(\gamma) = f_n(\rho_n)$ is non-negative and measurable. If $f_n(\rho_n)$ is integrable for each $n\in\mathcal{N}$ and the series on the right side of (A2.1) converges then $f(\gamma)$ is integrable and (A2.1) is valid:

It can be shown that* there exists a non-decreasing sequence $f_1, f_2, \ldots$

---

*See [6], p. 155 .

of non-negative simple functions such that for every $\gamma \in \Gamma$ $\lim_{k\to\infty} f_k(\gamma) = f(\gamma)$.

Then $\lim_{k\to\infty} f_k(\rho_n) = f(\rho_n) = f_n(\rho_n)$. If $f_n(\rho_n)$ is integrable then

$$\text{(A2.5)} \qquad \lim_{k\to\infty} \int_{P_n} f_k(\rho_n)d\mu_n = \int_{P_n} f_n(\rho_n)d\mu_n \ .$$

Now if the series converges we may reverse the steps in (A2.4). By making use of assertion 4 we see that $f(\gamma)$ is integrable and (A2.1) is valid.

7. If $f(\gamma)$ is integrable on $(\Gamma, \mathcal{A}, m)$ then $f_n(\rho_n) = f(\rho_n)$ is integrable on $(P_n, \mathcal{L}_n, \mu_n)$ for each $n \in \mathcal{N}$, equation (A2.1) holds, and the series converges absolutely:

If f is integrable, then by definition f^+ and f^- are integrable. Hence $f_n^+(\rho_n)$ and $f_n^-(\rho_n)$ are integrable for every $n \in \mathcal{N}$. By definition,

$$\text{(A2.6)} \qquad \int_\Gamma f dm = \int_\Gamma f^+ dm - \int_\Gamma f^- dm = \sum_{n\in\mathcal{N}} \int_{P_n} f^+(\rho_n)d\mu_n - \sum_{n\in\mathcal{N}} \int_{P_n} f_n^-(\rho_n)d\mu_n \ .$$

Since both series converge, the right side of this equation converges absolutely and can be rearranged. Thus

$$\text{(A2.7)} \qquad \int_\Gamma f dm = \sum_{n\in\mathcal{N}} \left[\int_{P_n} f_n^+(\rho_n)d\mu_n - \int_{P_n} f_n^-(\rho_n)d\mu_n \right] = \sum_{n\in\mathcal{N}} \int_{P_n} f_n(\rho_n)d\mu_n \ .$$

8. T_t is measure-preserving with respect to $(\Gamma, \mathcal{A}, m)$, for $t \geq 0$:

For any $A \in \mathcal{A}$, $(T_t^{-1}A) \cap P_n = T_t^{-1}(A \cap P_n) = S_{nt}^{-1}(A \cap P_n) \in \mathcal{L}_n$ for every $n \in \mathcal{N}$. Hence $T_t^{-1} A \in \mathcal{A}$.

$$\text{(A2.8)} \qquad m(T_t^{-1} A) = \sum_{n\in\mathcal{N}} \mu_n \left[(T_t^{-1}A) \cap P_n \right] = \sum_{n\in\mathcal{N}} \mu_n \left[S_{nt}^{-1}(A \cap P_n) \right]$$

$$= \sum_{n\in\mathcal{N}} \mu_n(A \cap P_n) = m(A) \ .$$

9. $T_{s+t} = T_s T_t$, for $t \geq 0$, $s \geq 0$:

This follows immediately from the same assertion for the S_{nt}.

10. $T_t\gamma$ is a measurable function of t and γ:

For each $n \in \mathcal{N}$, $S_{nt}\rho_n$ is a measurable function of t and ρ_n, i.e. $S_{nt}\rho_n$ is a measurable transformation from $(\mathcal{T} \times P_n, \mathcal{M} \times \mathcal{L}_n)$ into $(P_n, \mathcal{L}_n)$. Here $\mathcal{M}$ denotes the class of Lebesque measurable sets in $\mathcal{T} = \{t \ni 0 \leq t\}$. Thus for any $L_n \in \mathcal{L}_n$ $\{(t,\rho_n) \ni S_{nt}\rho_n \in L_n\} \in \mathcal{M} \times \mathcal{L}_n$. Now, given any $A \in \mathcal{A}$, $\{(t,\gamma) \ni T_t\gamma \in A\} = \underset{n\in\mathcal{N}}{U} \{(t, \rho_n) \ni T_t\rho_n \in A\} = \underset{n\in\mathcal{N}}{U} G_n$, where $G_n = \{(t,\rho_n) \ni S_{nt}\rho_n \in A \cap P_n\}$. For each n, G_n is a set in the class $\mathcal{M} \times \mathcal{L}_n$. But $\mathcal{L}_n \subset \mathcal{A}$, hence $\mathcal{M} \times \mathcal{L}_n \subset \mathcal{M} \times \mathcal{A}$. Thus, for each n, $G_n \in \mathcal{M} \times \mathcal{A}$, hence $\{(t,\gamma) \ni T_t\gamma \in A\} \in \mathcal{M} \times \mathcal{A}$, i.e. $T_t\gamma$ is a measurable transformation from $(\mathcal{T} \times \Gamma, \mathcal{M} \times \mathcal{A})$ into $(\Gamma, \mathcal{A})$.

11. $(\Gamma, \mathcal{A}, m, T_t)$ is a measure-preserving space.

This follows at once from 1,2,8,9 and 10.

12. $g(\gamma)$ is invariant with respect to $(\Gamma, \mathcal{A}, m, T_t)$:

For each $t \geq 0$, $g(T_t\gamma) = g_{(n)}(S_{nt}\rho_n) = g_{(n)}(\rho_n) = g(\gamma)$ except on a set $S \subset \Gamma$ such that $\mu_n(S \cap P_n) = 0$ for each $n \in \mathcal{N}$. Thus $m(S) = 0$.

13. $n(\gamma)$ is invariant with respect to $(\Gamma, \mathcal{A}, m, T_t)$:

$n(T_t\gamma) = n(S_{nt}\rho_n) = n = n(\gamma)$.

14. $y(\gamma) = [n(\gamma), g(\gamma)]$ is a complete invariant vector on $(\Gamma, \mathcal{A}, m, T_t)$:

Let $h(\gamma)$ be any invariant function, i.e. $h(T_t\gamma) = h(\gamma)$ a.e. $[m]$. For each fixed n, $h(S_{nt}\rho_n) = h(\rho_n)$ a.e. $[\mu_n]$. Hence, since $g_{(n)}$ is a complete invariant vector on $(P_n, \mathcal{L}_n, \mu_n, S_{nt})$, $h(\rho_n) = H_n[g_{(n)}(\rho_n)]$ a.e. $[\mu_n]$, where H_n is Borel-measurable on R^k. Let $z = (z_1, \ldots, z_\delta)$ be a point in R^δ and set

$$(A2.9) \qquad H(z,g) = \left\{ \begin{array}{ll} H_n(g) & \text{if } z = n \in \mathcal{N} \\ 0 & \text{if } z \notin \mathcal{N} \end{array} \right\} .$$

We shall show that H is Borel-measurable on $R^{k+\delta}$. If B is any Borel set in R^1 which does not contain the origin, then

$$(A2.10) \quad H^{-1}(B) = \left\{(z,g) \ni H(z,g) \in B\right\} = \bigcup_n \left\{(z,g) \ni H(z,g) \in B;\ z = n\right\}$$

$$= \bigcup_n \left\{(z,g) \ni H_n(g) \in B;\ z = n\right\} = \bigcup_n \left\{(z,g) \ni g \in H_n^{-1}(B);\ z = n\right\}$$

$$= \bigcup_n \left\{ z = n \right\} \times H_n^{-1}(B) .$$

The set $\{z = n\}$ is just a point in R^{δ}, hence is a Borel set. $H_n^{-1}(B)$ is also a Borel set in R^k. Hence $H^{-1}(B)$ is a Borel set in $R^{k+\delta}$. Now let B_o be the (Borel) set consisting of the origin in R^1.

$$(A2.11) \quad H^{-1}(B_o) = \left\{(z,g) \ni z \notin \mathcal{N}\right\} \cup \bigcup_n \left\{(z,g) \ni H(z,g) = 0;\ z = n\right\}$$

The set $\{(z,g) \ni z \notin \mathcal{N}\}$ is a Borel set in $R^{k+\delta}$, and for each n

$$(A2.12) \quad \left\{(z,g) \ni H(z,g) = 0;\ z = n\right\} = \left\{(z,g) \ni H_n(g) = 0;\ z = n\right\}$$

$$= \left\{(z,g) \ni g \in H_n^{-1}(B_o);\ z = n\right\} = \left\{ z = n \right\} \times H_n^{-1}(B_o) .$$

But this is again a Borel set, hence $H^{-1}(B_o)$ is a Borel set in $R^{k+\delta}$, and it follows that H is Borel-measurable on $R^{k+\delta}$. Now

(A2.13) $h(\gamma) = h(\rho_n) = H_n\,[g_{(n)}(\rho_n)] = H\,[n, g_{(n)}(\rho_n)] = H\,[n(\gamma), g(\gamma)]$,

except on a set such that $\mu_n(A \cap P_n) \neq 0$ for each $n \in \mathcal{N}$. Hence $h(\gamma) = H[n(\gamma), g(\gamma)]$ a.e. $[m]$.

15. $(\Gamma, \mathcal{A}, m, T_t, \gamma)$ is a complete space.

This follows directly from 11 and 14.

References

[1] Grad, H. New York University Lecture Notes (1950)

[2] Grad, H. Statistical Mechanics, Thermodynamics, and Fluid Dynamics of Systems with an Arbitrary Number of Integrals; Comm. Pure Appl. Math.; 5 , 455 - 494(1952).

[3] Halmos, P.R. Measure Theory; N.Y., D. Van Nostrand Co., Inc. (1950).

[4] Halmos, P.R. Lectures on Ergodic Theory, The Mathematical Society of Japan (1956).

[5] Khinchin, A.I. Mathematical Foundations of Statistical Mechanics; N.Y., Dover Publications, Inc. (1949) .

[6] Munroe, M.E. Introduction to Measure and Integration, Reading, Mass., Addison-Wesley Publishing Co., Inc. (1953).

DISTRIBUTION LIST FOR RESEARCH REPORTS

Contract No. AF 49(638)-341

(ONE copy unless otherwise noted)

(10) Commander
Air Force Office of Scientific Research
Air Research and Development Command
ATTN: SRY
Washington 25, D.C.

(4) Commander
Wright Air Development Center
ATTN: WCOSI-3
Wright-Patterson Air Force Base, Ohio

Commander
Air Force Cambridge Research Center
ATTN: Technical Library
L.G. Hanscom Field
Bedford, Mass.

Commander
Rome Air Development Center
ATTN: RCSST-4
Griffiss Air Force Base
Rome, N.Y.

Director, Office for Advanced Studies
Air Force Office of Scientific Research
Air Research and Development Command
P.O. Box 2035
Pasadena 2, California

(2) Commander
European Office
Air Research and Development Command
47 rue Cantersteen
Brussels, Belgium

(10) Commander
ASTIA
ATTN: TIPDR
Arlington Hall Station
Arlington 12, Virginia

Director of Research and Development
Headquarters USAF
ATTN: AFDRD
Washington 25, D.C.

Chief of Naval Research
Department of the Navy
ATTN: Code 420
Washington 25, D.C.

Director, Naval Research Laboratory
ATTN: Technical Information Officer
Washington 25, D.C.

Director, Research and Development Division
General Staff
Department of the Army
Washington 25, D.C.

Chief, Physics Branch, Division of Research
U.S. Atomic Energy Commission
1901 Constitution Ave., N.W.
Washington 25, D.C.

U.S. Atomic Energy Commission
Technical Information Extension
P.O. Box 62
Oak Ridge, Tennessee

National Bureau of Standards Library
Room 203, Northwest Building
Washington 25, D.C.

National Science Foundation
1520 H Street, N.W.
Washington 25, D.C.

Director, Office of Ordnance Research
Box CM, Duke Station
Durham, North Carolina

Office of Technical Services
Department of Commerce
Washington 25, D.C.

Commander
Arnold Engineering Development Center
Tullahoma, Tennessee
ATTN: Technical Library

Commander
Air Force Armament Center
Eglin Air Force Base, Florida
ATTN: Technical Library

Commander
Air Force Flight Test Center
U.S. Air Force
ATTN: FTOTL
Edwards Air Force Base, California

Commander
Air Force Missile Test Center
ATTN: Technical Library
Patrick Air Force Base
Cocoa, Florida

Commander
Air Force Special Weapons Center
ATTN: Technical Library
Kirtland Air Force Base, New Mexico

Commander
Air Force Missile Development Center
ATTN: Technical Library
Holloman Air Force Base, New Mexico

Commander
Air Force Cambridge Research Center
ATTN: CRQOTR
L.G. Hanscom Field
Bedford, Mass.

Commander
Army Rocket and Guided Missile Agency
Redstone Arsenal
ATTN: ORDXR-OTL
Alabama

Commandant
Air Force Institute of Technology
ATTN: MCLI, Technical Library
Wright-Patterson Air Force Base, Ohio

Commander
Air Force Ballistic Missile Div., HQ, ARDC
ATTN: WDSOT
P.O. Box 262
Inglewood, California

Applied Mechanics Review
Southwest Research Institute
8500 Culebra Rd.
San Antonio, Texas

Institute of the Aeronautical Sciences
ATTN: Librarian
2 E. 64th St.
New York 16, N.Y.

NOV 1 9 1959

Date Due

MAY 14 '62

DEC 21 '67

JAN 9 '68

OCT 7 1974

OCT 19 1974

NOV 22 1974

PRINTED IN U. S. A.

NYU
HT-
2
AUTHOR
Lewis, R. M.

NYU
HT-
2
AUTHOR
Lewis, R. M.
TITLE
Measure-theoretic foundations of statistical mechanics.

DATE DUE | BORROWER'S NAME | ROOM NUMBER
MAY 10 '62 | Magnus
DEC 21 '6
JAN 9 1968 RENEWED

N. Y. U. Institute of
Mathematical Sciences
25 Waverly Place
New York 3, N. Y.